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Approximation by Minimum Norm Interpolants in Uniform Algebras

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We prove here the

THEOREM. *Let A be a uniform algebra on X, $E \subset X$ a generalized peak set for A, and $f \in A$. Then for every $\varepsilon > 0$ there exists $g \in A$ satisfying:*

(i) $g|E = f|E$;

(ii) $\|g\|_X = \|f\|_E$; *and*

(iii) $\|g - f\|_X < \|f\|_X - \|f\|_E + \varepsilon$.

This both refines and extends the result of [1] in which the theorem is proved for A the disc algebra on the unit circle T, f a fixed member of A, and E a varying closed subset of T of Lebesgue measure 0, and condition (iii) is replaced by $\|g - f\|_T = 0(\|f\|_T - \|f\|_E)$; it is also proved in [1] that 0 cannot be replaced by o, a fact which is immediate from condition (ii).

The theorem is sharp in that the ε can never be removed in (iii), and even for the disc algebra one cannot ordinarily replace (iii) by $\|g - f\|_T \leqslant \|f\|_T - \|f\|_E$: if $f \in A$ is not of constant modulus but assumes its maximum modulus throughout some arc of T, take E to consist of one point at which $|f|$ is strictly less than this maximum.

The method of proof in this paper is rather different from that in [1]. We use duality together with the following fact [2, II. 12.7 and II.12.5] and its consequence:

If $u \in A|E$ and p is a positive continuous function on X such that $|u| \leqslant p$ on E, then there is $w \in A$ such that $w = u$ on E and $|w| \leqslant p$ on X. (1)

If $g \in A$ and G is any G_δ set containing E, then there is a sequence (g_n) in A such that $g_n \equiv 0$ on E, $\|g_n\|_X \leqslant \|g\|_X$, and $g_n \to g$ pointwise off G. (2)

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To derive (2) from (1), use (1) to obtain a sequence (w_n) in A such that $w_n \equiv 1$ on E, $\|w_n\|_X = 1$, and $w_n \to 0$ pointwise off G; if (λ_n) is a sequence of numbers such that $|\lambda_n| < 1$ but $\lambda_n \to -1$, then $g_n = ((\lambda_n w_n - \lambda_n)(1 - \bar{\lambda}_n(\lambda_n w_n))^{-1})g$ works in (2). This is the only point at which the algebra structure of A plays a direct role. If A is merely a closed subspace of $C(X)$, then [2, II.12.5], (1) is equivalent to having $\mu_E \in A^\perp$ whenever $\mu \in A^\perp$. In this situation, (1) permits us to find (w_n) in A such that $w_n = g$ on E, $\|w_n\|_X \leqslant \|g\|_E$, and $w_n \to 0$ pointwise off G, and $g_n = g - w_n$ allows us to deduce (2) with $\|g_n\|_X \leqslant 2\|g\|_X$. The net result is that the theorem remains true with (iii) replaced by $\|g - f\|_X < 2(\|f\|_X - \|f\|_E) + \varepsilon$.

We now prove the theorem. We may suppose $\|f\|_E = 1 < \rho = \|f\|_X$. Denote by B the closed unit ball of A and by B_E the set of functions in B which vanish identically on E. We begin by proving that if $c > 1$ then

$$cB \cap (f + (\rho - 1)B_E) \neq \varnothing. \qquad (3)$$

If (3) were false there would be ϕ in the closed unit ball of A^* such that $|\phi(f+h)| \geqslant c$ for all $h \in (\rho - 1)B_E$. ϕ is given by integration against a complex regular Borel measure μ of total variation norm $\|\mu\| = \|\phi\|$. It follows that if F is any Borel set then the functional on A induced by μ_F, the restriction of μ to F, has norm equal to $\|\mu_F\|$, the total variation of μ_F. Let G be a G_δ set containing E such that μ_{E^c} is supported off G. It follows from (2) and our observations that $\sup\{|\phi(h)|: h \in B_E\} = \|\mu_{E^c}\|$. Since $|\phi(f)| \leqslant \|\mu_E\| + \rho\|\mu_{E^c}\| \leqslant 1 + (\rho - 1)\|\mu_{E^c}\|$ we have $\inf\{|\phi(f+h)|: h \in (\rho - 1)B_E\} \leqslant |\phi(f)| - \sup\{|\phi(h)|: h \in (\rho - 1)B_E\} \leqslant 1 + (\rho - 1)\|\mu_{E^c}\| - (\rho - 1)\|\mu_{E^c}\| = 1$, contradicting our choice of ϕ.

Thus (3) holds for $c > 1$, so we may choose $g_c \in c^{-1}B \cap (c^{-2}f + c^{-2}(\rho - 1)B_E)$. Define a function p on X by $p = \min(1 - c^{-2}, 1 - |g_c|)$. p is continuous, $p \geqslant 1 - c^{-1} > 0$ on X, and on E $|(1 - c^{-2})f| \leqslant 1 - c^{-2} \leqslant 1 - |g_c|$ so that $|(1 - c^{-2})f| \leqslant p$. By (1) there is $h_c \in A$ such that $h_c = (1 - c^{-2})f$ on E and $|h_c| \leqslant p$ on X. Then $g = g_c + h_c$ clearly satisfies (i) and (ii), and

$$\begin{aligned}\|f - g\|_X &\leqslant \|f - c^{-2}f\|_X + \|c^{-2}f - g_c\|_X + \|h_c\|_X \\ &\leqslant (1 - c^{-2})\rho + c^{-2}(\rho - 1) + (1 - c^{-2})\end{aligned}$$

which is less than $\rho - 1 + \varepsilon$ provided c is near enough to 1 so that $c^2 < 2/(2 - \varepsilon)$. The proof is complete.

References

1. C. K. Chui, W. Smith, and J. D. Ward, Approximation by minimum norm interpolants in the disc algebra, *J. Approx. Theory* **27** (1979), 291–295.
2. T. W. Gamelin, "Uniform Algebras," Prentice–Hall, Englewood Cliffs, N. J., 1969.

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